Title	JC and Polytechnic Mathematics Compilation Series – Pure Mathematics
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Title	JC and Polytechnic Mathematics – Basic Operation of Complex Numbers
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Date	8/12/2018

This portion is designed to address a basics of complex numbers addition, subtraction, multiplication and division and does not involve any geometry.

Applicable to

- JC 'A' Level H2 Mathematics
- Nanyang Polytechnic, School of Engineering – Engineering Mathematics 1A
- Nanyang Polytechnic, School of Information Technology – Diploma Plus Program (Mathematics)

Introducing the imaginary unit i (Engineering students use j instead)
$i = \sqrt{-1}$

If you have read my document on secondary school algebra, you will realize that I specifically tell students not to use i in any algebra operations (unless question ask you to do so, of course) as it is reserved for specific use. For a long time, you may be thinking what exactly it is being used for, it is in fact used to represent the value of $\sqrt{-1}$ at higher level mathematics.

Basic Operation of Imaginary Numbers
Addition Rule
$xi + yi = (x + y)i$

Subtraction Rule
$xi - yi = (x - y)i$

Multiplication Rule
$i(i) = -1$
$(-i)(i) = 1$
$xi\,(yi) = x(y)(-1) = -xy$

Division Rule
$$\frac{xi}{yi} = \frac{x}{y}$$

Basic Examples:

Q1. $6i + 8i$

Q2. $5i - 3i$

Q3. $3i(4i)$

Q4. $\frac{2i}{5i}$

Q1	$6i + 8i = 14i$
Q2	$5i - 3i = 2i$
Q3	$3i(4i) = -12$
Q4	$\frac{2i}{5i} = \frac{2}{5}$

If a number contains both an imaginary and a real number, the number is called a complex number.

A complex number is in the following form

$$a + bi$$

a is the real part, bi is the imaginary part.

Complex Number Operations

Addition Rule

$(a + bi) + (g + hi) = (a + g) + (b + h)i$

Subtraction Rule

$(a + bi) - (g + hi) = (a - g) + (b - h)i$

Multiplication Rule (I hope you recall your secondary school expansion and factorization class, as it is similar to that, with one exception, that $i(i) = -1$)

$(a + bi)(g + hi) = ag + bgi + ahi + (-1)bh$

$ag - bh + bgi + ahi = (ag - bh) + (bg + ah)i$

Division Rule (Students who took Additional Mathematics, this may ring a bell as it is similar to the method you used to rationalize denominator in secondary 3.)

Situation 1.

$\frac{a + bi}{g + hi} = \frac{a + bi}{g + hi} \times \frac{g - hi}{g - hi}$

Situation 2.
$$\frac{a+bi}{g-hi}=\frac{a+bi}{g-hi}\times\frac{g+hi}{g+hi}$$

This method is called conjugate multiplication, a conjugate has the following properties that makes the above method work.

1. The conjugate multiplied to the complex number will always be equal to 1.
2. The signs between the imaginary part and the real part of the denominator has to flip to become the conjugate, (i.e. "+" to "-" and "+" to "+")

Basic Examples
Q5. $(3-6i)-(7-8i)$
Q6. $(15+6i)+(5-3i)$

Q7. $(5+i)(6-3i)$

Q8. $\frac{2+5i}{1+6i}$

Q5	By distributive law the expression is equal to the following $(3-6i)-(7-8i)$ $=3-6i-7+8i$ $=-4+2i$
Q6	$(15+6i)+(5-3i)$ $=20+3i$
Q7.	Like how you expand expression in brackets in secondary school, with an exception to note that $i\times i=-1$ $(5+i)(6-3i)$ $=30+6i-15i+(i)(-3i)$ $=30-9i+(-1)(-3)$ $=30+3-9i$ $=33-9i$

Q8.	The conjugate of the expression $\frac{2+5i}{1+6i}$ is $1-6i$ $\frac{2+5i}{1+6i}\times\frac{1-6i}{1-6i}$ $=\frac{(2+5i)(1-6i)}{(1+6i)(1-6i)}$ $=\frac{2+5i-12i+30}{1-36(i)(i)}$ $=\frac{32-7i}{1-36(-1)}$ $=\frac{32-7i}{1+36}$ $=\frac{32-7i}{37}$

Title	Complex Numbers – Argument and Modulus [Converting from Algebraic Form to Polar and Exponential Form Form]
Author	Lim Wang Sheng, School of Information Technology, Nanyang Polytechnic [CCA: NYP Mentoring Club]
Date	15/12/2018

Applicable to

- Junior College 'A' Level H2 Mathematics
- Nanyang Polytechnic, School of Engineering, Engineering Mathematics 1A
- Nanyang Polytechnic, School of Information Technology, Mathematics (Diploma Plus Program)

Basic Notation	
r	The modulus of the complex number. (Also denoted in textbooks using the 2 vertical strokes. If z is the complex number, the modulus can be written as $\lvert z \rvert$.
$\theta_{argument}$	The argument of the complex numbers. (Denoted in textbooks as $\arg(z)$, given z is the complex number.)

Notation Example	
$a + bi$ (Algebraic Form)	Algebraic Interpretation: a is the real part of the complex number and b is the imaginary part of the complex number.
$r\angle\theta$ (Polar Coordinate Form)	Given complex number $a + bi$ r refers to the distance of the point on the complex plane relative to the origin point of the complex plane. Can be computed using $r = \sqrt{a^2 + b^2}$ (Also known as the modulus) θ refers to angle of the line relative the positive real-axis, that start from the point of origin to the coordinate of $a + bi$ (Also known as the argument)
$r(\cos(\theta) + i\sin(\theta))$ (Polar Form)	Given complex number $a + bi$ r refers to the distance of the point on the complex plane relative to the origin point of the complex plane. Can be computed using $r = \sqrt{a^2 + b^2}$ (Also known as the modulus)

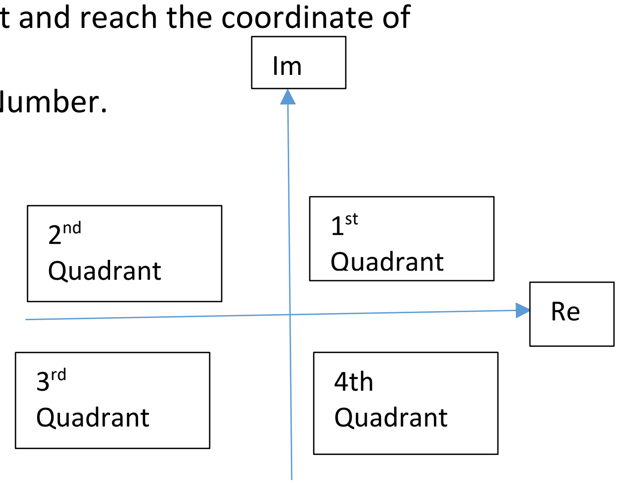
	θ refers to angle of the line relative the positive real-axis, that start from the point of origin to the coordinate of $a + bi$ (Also known as the argument)
$re^{i\theta}$ (Exponential Form)	Given complex number $a + bi$ r refers to the distance of the point on the complex plane relative to the origin point of the complex plane. Can be computed using $r = \sqrt{a^2 + b^2}$ (Also known as the modulus) θ refers to angle of the line relative the positive real-axis, that start from the point of origin to the coordinate of $a + bi$ (Also known as the argument)

As you can see the required values for polar coordinate, polar form and exponential form are computed using the same formula, so I will just need to guide you all on how to compute the modulus and argument.

<u>(Make sure your calculator is in “Radian Mode” before proceeding)</u>

Step 1. Sketch a complex plane and plot the coordinate of the complex number within the complex plane. (This is also called an argand diagram.)
Step 2. Draw a line, the line shall start from the origin point and reach the coordinate of the complex number.
Step 3. Determine the Quadrant Number of the Complex Number.

(Im means imaginary axis, Re means the real axis.)



Step 4. Compute Basic Angle of the Complex Number
Using $\theta_{basic} = \tan^{-1}\left(\frac{Opposite}{Adjacent}\right)$

Step 5. Use the following rules to deduce the argument
For first quadrant
$\theta_{argument} = \theta_{basic}$

For second quadrant
$\theta_{argument} = \pi - \theta_{basic}$

For third quadrant
$\theta_{arguement} = -\pi + \theta_{basic}$

For fourth quadrant
$\theta_{argument} = -\theta_{basic}$

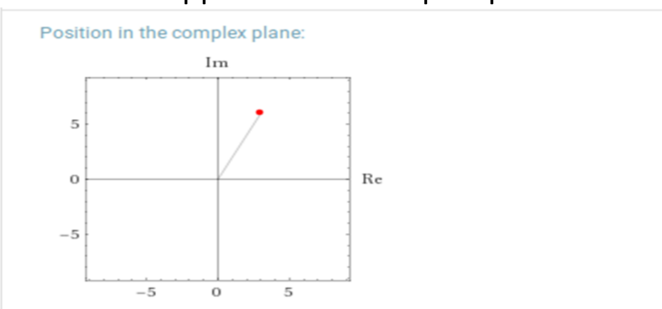
Step 6. Compute r using the below formula
$r = \sqrt{a^2 + b^2}$

Find the modulus and argument of the following complex numbers.

1. $3 + 6i$
2. $1 - 3i$
3. $-1 + 3i$
4. $-1 - i$

Question 1.
First quadrant therefore:
$\theta_{argument} = \theta_{basic}$

$3 + 6i$ as mapped on the complex plane

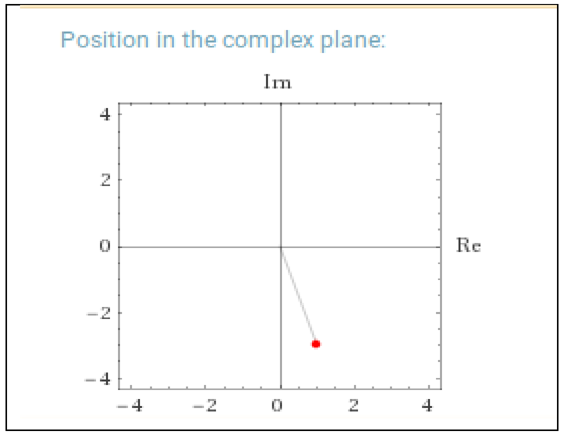


$\theta = \tan^{-1}\left(\frac{6}{3}\right) = 1.11\ Radians$

$r = \sqrt{6^2 + 3^2} = \sqrt{36 + 9} = \sqrt{45} = 3\sqrt{5}$

Written in Polar Coordinate Form: $3\sqrt{5} \angle 1.11$
Written in Polar Form $3\sqrt{5}\ (\cos(1.11) + i\sin(1.11))$
Written in Exponential Form $3\sqrt{5}e^{i\ (1.11)}$

Question 2.
$1 - 3i$ as mapped on the complex plane.



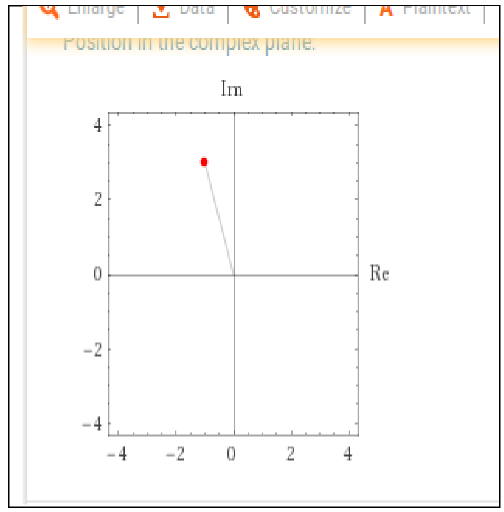
The complex position of the value lies in the fourth quadrant of the complex plane, therefore, $\theta_{argument} = -\theta_{basic}$
In this case, the opposite is the absolute value of the imaginary part while the adjacent is the real part of the complex number.

*When dealing with angles, always convert coordinate values to absolute values.

$$\theta_{basic} = \tan^{-1}\left(\frac{3}{1}\right) = 1.249\ Radians$$
$$\theta_{argument} = -1.249\ Radians$$
$$r = \sqrt{1^2 + (-3)^2} = \sqrt{10}$$

Written in Polar Coordinate Form:	$\sqrt{10}\angle - 1.25 Radians$
Written in Polar Form	$\sqrt{10}(\cos(-1.25) + i\sin(-1.25))$
Written in Exponential Form	$\sqrt{10}\, e^{-1.25\, i}$

Question 3.
$-1+3i$ as mapped on the complex plane



Since in Second Quadrant, the Argument is computed as follows

$\theta_{argument} = \pi - \theta_{basic}$

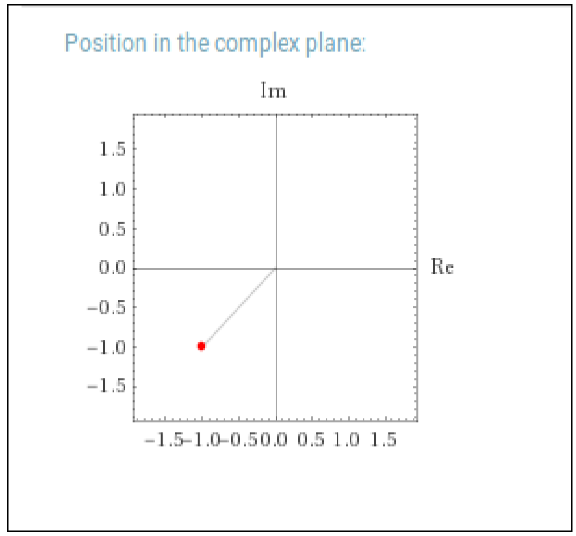
$\theta_{basic} = \tan^{-1}\left(\frac{3}{1}\right) = 1.24905\ Radians$

$\theta_{argument} = \pi - 1.24905\ Radians = 1.89\ Radians$

$r = \sqrt{3^2 + 1^2} = \sqrt{10}$

Written in Polar Coordinate Form	$\sqrt{10}\ \angle 1.89\ Radians$
Written in Polar Form	$\sqrt{10}\ (\cos(1.89) + i\sin(1.89))$
Written in Exponential Form	$\sqrt{10}(e^{1.89i})$

Question 4.
$-1 - i$ as mapped on the complex plane.



Since the coordinate of the complex number lies in the third quadrant, we use the following formula

$\theta_{arguement} = -\pi + \theta_{basic}$

$\theta_{basic} = \tan^{-1}\left(\frac{1}{1}\right) = 0.78540\ Radians$

$\theta_{arguement} = -\pi + 0.78540 = -2.36\ Radians$

$r = \sqrt{1^2 + 1^2} = 1.4142$

Written in Polar Coordinate Form	$\sqrt{2}\angle -2.36\ Radians$
Written in Polar Form	$\sqrt{2}\,(\sin(-2.36) + i\sin(-2.36))$
Written in Exponential Form	$\sqrt{2}\,(e^{-2.36i})$

Title	Differentiation of Inverse Trigonometric Functions
Author	AprilDolphin
Date	19/9/2024
Assumptions	This article assumes you already have good knowledge of using chain rule to differentiate functions as well as basic knowledge of finding derivatives of algebraic, logarithmic, exponential and trigonometric functions.

Function	Derivatives
$f(x) = \sin^{-1}(x)$	$f'(x) = \frac{1}{\sqrt{1-x^2}}$
$f(x) = \cos^{-1}(x)$	$f'(x) = \frac{-1}{\sqrt{1-x^2}}$
$f(x) = \tan^{-1}(x)$	$f'(x) = \frac{1}{1+x^2}$

Chain Rule in General
Given $f(x) = g\big(h(x)\big)$ $f'(x) = g'(x)\,h'(x)$

Application of Chain Rule to Inverse Trigonometric Functions	
Function	Derivatives
$f(x) = \sin^{-1}(u)$	$f'(x) = \frac{1}{\sqrt{1-u^2}}(u')$
$f(x) = \cos^{-1}(u)$	$f'(x) = \frac{-1}{\sqrt{1-u^2}}(u')$
$f(x) = \tan^{-1}(u)$	$f'(x) = \frac{1}{1+u^2}(u')$

Example 1.
Find the derivative of the following functions

(a) $\mathrm{y} = \sin^{-1}(e^{6t})$

Using chain rule $\frac{dy}{dt} = \frac{dy}{du} \times \frac{du}{dt}$

$$\frac{dy}{du} = \frac{1}{\sqrt{(1-u^2)}} \times 6e^{6t} = \frac{6e^{6t}}{\sqrt{1-e^{12t}}}$$

(b) $y = \tan^{-1}(2^x)$

Using chain rule $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$$\frac{1}{1+(2^x)^2} \times 2^x \ln(2) = \frac{2^x \ln(2)}{1+2^{2x}}$$

(c) $f(x) = \cos^{-1}\left(\frac{x}{3}\right)$

Using chain rule, where any composite function $f(x) = g\big(h(x)\big)$ will produce a derivative of $g'(x)h'(x)$.

$$\cos^{-1}\left(\frac{x}{3}\right) = \cos^{-1}\left(\frac{1}{3}x\right)$$

$$f'(x) = \frac{-1}{\sqrt{1-\left(\frac{1}{3}x\right)^2}} \times \frac{1}{3} = -\frac{1}{3\sqrt{1-\frac{1}{9}x^2}} = -\frac{1}{\sqrt{9}\sqrt{1-\frac{x^2}{9}}}$$

$$= -\frac{1}{\sqrt{9-\frac{\sqrt{9^2}x^2}{9}}} = -\frac{1}{\sqrt{9-x^2}}$$

(d) $y = e^{\sin^{-1}(2x)}$

Using chain rule, where $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$$2e^{\sin^{-1}(2x)} \times \frac{1}{\sqrt{1-(2x^2)}} = \frac{2e^{\sin^{-1}(2x)}}{\sqrt{1-4x^2}}$$